

Limits to Infinity

In this handout we will examine limits of the form

$$\lim_{x \rightarrow \infty} \frac{g(x)}{h(x)} \quad \text{or} \quad \lim_{x \rightarrow -\infty} \frac{g(x)}{h(x)}$$

where $g(x)$ and $h(x)$ are polynomials.

The important thing to remark is that we are taking the limit to ∞ or $-\infty$ of a rational function. We note that

$$\lim_{x \rightarrow \infty} \frac{\text{Any finite number}}{x^n} = 0 \quad \text{or} \quad \lim_{x \rightarrow -\infty} \frac{\text{Any finite number}}{x^n} = 0 \quad \text{for } n \in \mathbb{N}.$$

Using the above facts, we try to rewrite our functions using terms like $\frac{c}{x^n}$ where $c \in \mathbb{R}$. To do this we divide every term in the numerator and denominator by the highest power of x that appears in the function.

Example 1

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2 - 9x + 8}{5x^2 - x} &= \lim_{x \rightarrow \infty} \frac{\frac{x^2}{x^2} - \frac{9x}{x^2} + \frac{8}{x^2}}{\frac{5x^2}{x^2} - \frac{x}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{1 - \frac{9}{x} + \frac{8}{x^2}}{5 - \frac{1}{x}}. \end{aligned}$$

We are now in a position to take the limit. We get

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{1 - \frac{9}{x} + \frac{8}{x^2}}{5 - \frac{1}{x}} &= \frac{1 - 0 + 0}{5 - 0} \\ &= \frac{1}{5}. \end{aligned}$$

In summary, we have found that

$$\lim_{x \rightarrow \infty} \frac{x^2 - 9x + 8}{5x^2 - x} = \frac{1}{5}.$$

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Example 2

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{3x^2 - 6x + 2}{5x^3 - 2x^2 + 6x + 7} &= \lim_{x \rightarrow -\infty} \frac{\frac{3x^2}{x^3} - \frac{6x}{x^3} + \frac{2}{x^3}}{\frac{5x^3}{x^3} - \frac{2x^2}{x^3} + \frac{6x}{x^3} + \frac{7}{x^3}} \\ &= \lim_{x \rightarrow -\infty} \frac{\frac{3}{x} - \frac{6}{x^2} + \frac{2}{x^3}}{5 - \frac{2}{x} + \frac{6}{x^2} + \frac{7}{x^3}} \\ &= \frac{0 - 0 + 0}{5 - 0 + 0 + 0} \\ &= 0.\end{aligned}$$

Example 3

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{4x^3 + 5}{5x^2 - 1} &= \lim_{x \rightarrow \infty} \frac{\frac{4x^3}{x^3} + \frac{5}{x^3}}{\frac{5x^2}{x^3} - \frac{1}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{4 + \frac{5}{x^3}}{\frac{5}{x} - \frac{1}{x^3}} \\ &= \infty.\end{aligned}$$

Try the following exercises for practice.

Calculate the following limits:

(a)

$$\lim_{x \rightarrow \infty} \frac{x^4 - 2x^2 - x}{2x^4 + 3x^3 + 4}$$

(b)

$$\lim_{x \rightarrow \infty} \frac{3x^3 - 2x^2 - 2}{x^5 - x^2 + 1}$$

(c)

$$\lim_{x \rightarrow \infty} \frac{3x^5 + 6x^2}{7x^4 - 3x^2 - 4}$$

(d)

$$\lim_{x \rightarrow -\infty} \frac{3x^6 + 4x^5 - x^3 - 2x^2 - 1}{5x^6 - 4x^3 + 3x + 2}$$

(e)

$$\lim_{x \rightarrow -\infty} \frac{5x^2 - 1}{3x^6 - 2x^2 - 7}$$

Solutions

(a) $\frac{1}{2}$

(b) 0

(c) ∞

(d) $\frac{3}{5}$

(e) 0